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# Stability of Lie Superalgebras and Branes

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**Chryssomalis Chryssomalakos**

*Instituto de Ciencias Nucleares  
Universidad Nacional Autónoma de México  
Apdo. Postal 70-543, 04510 México, D.F., MEXICO  
chryss@nuclecu.unam.mx*

**ABSTRACT:** The algebra of the generators of translations in superspace is unstable, in the sense that infinitesimal perturbations of its structure constants lead to non-isomorphic algebras. We show how superspace extensions remedy this situation (after arguing that remedy is indeed needed) and review the benefits reaped in the description of branes of all kinds in the presence of the extra dimensions.

# 1 Introduction

The aim of physical theories is the simplest possible description of Nature within the uncertainties afforded by experiments. The symmetry aspects of this description are usually codified in an underlying Lie algebra, around which the theory is built. The structure constants of the algebra parameterize the theory and are, in principle, measurable quantities — they often emerge as the fundamental constants of the theory. Given that our knowledge of these parameters can only be approximate, physical theories that do not change qualitatively under small perturbations of their parameters have more chances of wide applicability. This observation may not be worthy of an axiom status but it does provide a sensible criterion about which regions of parameter space might be most promising for theory hunting.

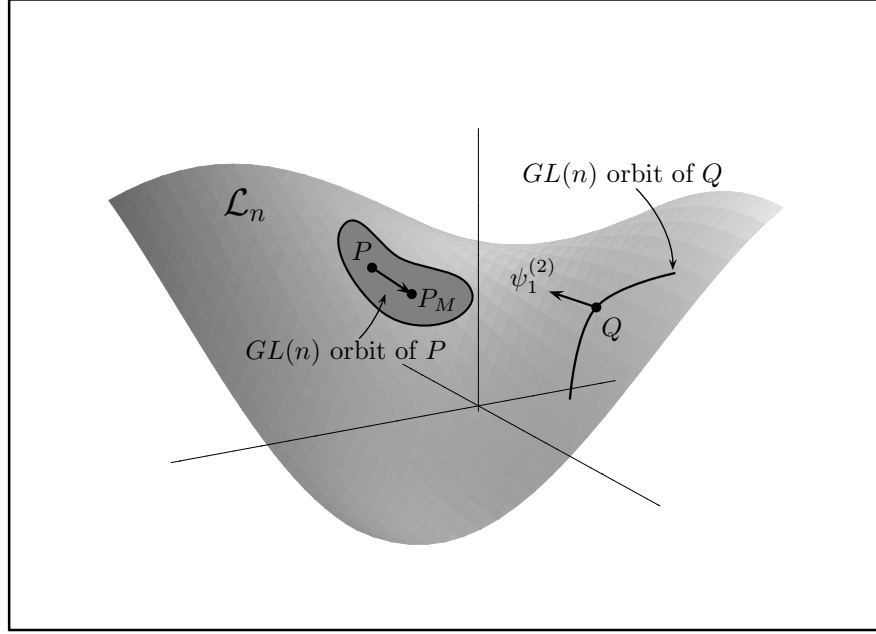
Lie algebras come in two varieties: *stable*<sup>1</sup> and *unstable*. The former are isomorphic to all Lie algebras in their vicinity (in a sense to be made precise later on) while the latter are infinitesimally close to qualitatively different (*i.e.* non-isomorphic) algebras. To specify the content of a physical theory relying on an unstable Lie algebra, an infinitely fine tuning of the values of its parameters is necessary. It makes therefore sense, when confronted with such a theory, to look for generalizations that are immune to perturbations, *i.e.*, are endowed with a definite structure despite the fuzziness in their parameters. This point of view has an already long history, and not only in physics. It permeates the work of Thom [20] on morphogenetic mechanisms in the seventies, which in turn draws on Smale's [19] strong advocacy, as early as in the sixties, of a similar principle in the study of non-linear dynamics. In physics, it has been pointed out by several authors [8, 7, 2, 12] that the kinematics of special relativity can be obtained as a stabilization of the Galilean algebra, an example that we also present in the sequel, as well as that the quantization of classical mechanics can be described as the transition from an (infinite dimensional) unstable Lie algebra of functions on phase space to a stable one. In either case, the mathematical process involved has been formalized in the theory of deformations of Lie algebras, developed in the sixties by Gersternhaber [9], Nijenhuis and Richardson [13] and others.

On a different front, alluded to by the second half of the title, recent years have witnessed an explosion of interest in extended objects, collectively referred to as branes, for which a variety of action functionals has by now been proposed and studied. The matching of the number of bosonic and fermionic degrees of freedom in these actions is guaranteed by the so-called  $\kappa$ -symmetry, which in turn necessitates the introduction of a WZ term. Already in the case of the superstring action of Green and Schwarz, a problem arises by the fact that the WZ term is not manifestly supersymmetric, but rather, transforms by a total derivative under general translations in superspace. Siegel [18] showed that by augmenting superspace by a new fermionic variable, one could write a truly invariant WZ term for the action, consisting of the pull-back on the worldvolume of an invariant 2-form defined on the *extended* superspace. At the algebra level this amounts to a central extension by a fermionic generator which transforms as a spinor under the Lorentz group. It was thus shown that, once extended objects are considered, further extensions of standard superspace might provide a natural background for the description of their dynamics. Following Siegel's work, Bergshoeff and Sezgin [3] proposed extended supersymmetry algebras that lend themselves to the description of  $p$ -branes. The zoo of extended objects (and of corresponding algebras) has been enriched since then with the appearance of  $D$ -branes [15] (and, later,  $L$ -branes [10]), the actions of which differ qualitatively from the standard brane action in that they involve fields defined directly on the worldvolume.

The plethora of extended objects mentioned above, combined with the disparity in the characteristics of their actions, motivated the systematic study of superalgebra extensions undertaken in [5]. It was shown there how genuinely invariant actions can be found on suitably extended superspaces, recovering the results of [3]. Even more tantalizingly, the Born-Infeld worldvolume fields were shown to come from pull-backs from these extended superspaces. Our aim here is to show how these results reinforce the particular point of view advocated above and to argue that they should be added to

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<sup>1</sup>the term *rigid* is often used by mathematicians as a synonym.



**Figure 1:** The space  $\mathcal{L}_n$  of  $n$ -dimensional Lie algebras.  $P$  is surrounded by equivalent points and hence,  $\mathcal{G}_P \sim \mathcal{G}_{P_M}$ , for all  $P_M$  sufficiently close to  $P$ . In contrast, in the tangent space of  $\mathcal{L}_n$  at  $Q$ , there are directions that lead outside of the orbit  $\mathcal{O}(Q)$ .  $Q$  will move along these directions when  $\psi_1^{(2)}$  in (4) is a non-trivial element of  $H^2(\mathcal{G}_Q)$ .

the list of developments illuminated by stability considerations. Sec. 2 starts with a cursory look at the formal deformation theory of Lie algebras, including the example of the Galilean algebra. Then branes enter, in Sec. 3, and some of the problems arising in the standard superspace formulation are pointed out. Sec. 4 shows how to extend superspace so that the resulting algebra of the generators of supertranslations is stabilized. Sec. 5 looks at the applications to standard  $p$ -branes and  $D$ -branes mentioned above. We contemplate on future possible directions, adhering to the stability theme, in the epilogue.

## 2 Deformations

Given an  $n$ -dimensional real<sup>2</sup> Lie algebra  $\mathcal{G}$ , with generators  $T_A$ ,  $A = 1, \dots, n$ .  $\mathcal{G}$  is specified completely by its (real) structure constants  $f_{AB}^C$ . The latter are subject to two conditions: antisymmetry in the lower two indices and the Jacobi identity

$$\begin{aligned} f_{AB}^C &= -f_{BA}^C, \\ f_{AR}^S f_{BC}^R + f_{BR}^S f_{CA}^R + f_{CR}^S f_{AB}^R &= 0. \end{aligned} \quad (1)$$

Relaxing for the moment the latter, one is left with  $N(n) = n^2(n-1)/2$  arbitrary constants. Consider now the space  $\mathbb{R}^N$ , with each of the  $f$ 's ranging along an axis. For each value of  $(A, B, C, S)$ , (1) describes a quadratic hypersurface in this space. The intersection of these hypersurfaces is the space  $\mathcal{L}_n$  of all possible  $n$ -dimensional Lie algebras — we sketch it as a surface in Fig. 1. Referring to this

<sup>2</sup>Lie algebra will always mean real Lie algebra in what follows.

figure, consider the point  $P$  on  $\mathcal{L}_n$  — it corresponds to the Lie algebra  $\mathcal{G}_P$ , whose structure constants are given by the coordinates of  $P$ . Under a linear redefinition of the generators via a  $GL(n)$  matrix  $M$ ,

$$T'_A = M_A{}^B T_B, \quad (2)$$

the structure constants transform as

$$f'^C_{AB} = M_A{}^R M_B{}^S (M^{-1})^C{}_U f^U_{RS}, \quad (3)$$

and  $P$  moves to  $P_M$ . Clearly, no new physics is to be expected from such a redefinition,  $\mathcal{G}_P$  and  $\mathcal{G}_{P_M}$  being isomorphic. What we are really interested in then, from a physical point of view, is not  $\mathcal{L}_n$  itself, but, rather, the equivalence classes into which  $\mathcal{L}_n$  splits under the above action of  $GL(n)$ , each class being the  $GL(n)$  orbit  $\mathcal{O}(P)$  of any point  $P$  in the class. The crucial observation to be made here is that there exist two types of points on  $\mathcal{L}_n$ : those that are completely surrounded by equivalent points and those whose neighborhoods include non-equivalent points, sketched as  $P$  and  $Q$  respectively in Fig. 1. Any infinitesimal perturbation of the structure constants of  $\mathcal{G}_P$  will necessarily lead to an isomorphic Lie algebra while there exist infinitesimal perturbations of  $\mathcal{G}_Q$  that lead outside of  $\mathcal{O}(Q)$  and, hence, to non-isomorphic algebras.

Given a Lie algebra  $\mathcal{G}$  with the Lie product of  $T_A, T_B \in \mathcal{G}$  supplied by the commutator  $[T_A, T_B]_0$ . A *one-parameter deformation* of  $\mathcal{G}$  is given by the *deformed commutator*

$$[T_A, T_B]_t = [T_A, T_B]_0 + \sum_{m=1}^{\infty} \psi_m^{(2)}(T_A, T_B) t^m, \quad (4)$$

where  $t$  is a formal parameter and the  $\psi_m^{(2)}$  are  $\mathcal{G}$ -valued, bilinear antisymmetric maps

$$\psi_m^{(2)} : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}, \quad \psi_m^{(2)}(T_A, T_B) = -\psi_m^{(2)}(T_B, T_A). \quad (5)$$

Such maps are called *2-cochains* — in a similar fashion one defines *p-cochains*  $\psi^{(p)}$  which accept  $p$  arguments. The *coboundary operator*  $s$  maps  $p$ -cochains to  $p+1$ -cochains according to

$$\begin{aligned} s \triangleright \psi^{(p)}(T_{A_1}, \dots, T_{A_{p+1}}) &= \sum_{r=1}^{p+1} (-1)^{p+1} [T_{A_r}, \psi^{(p)}(T_{A_1}, \dots, \hat{T}_{A_r}, \dots, T_{A_{p+1}})] \\ &+ \sum_{r < s} (-1)^{r+s} \psi^{(p)}([T_{A_r}, T_{A_s}], T_{A_1}, \dots, \hat{T}_{A_r}, \dots, \hat{T}_{A_s}, \dots, T_{A_{p+1}}). \end{aligned} \quad (6)$$

We will have more to say about  $s$  later on. For the moment, we just state that  $s^2 = 0$ . Then, mimicking a familiar procedure in differential geometry, one defines *p-cocycles* as  $p$ -cochains annihilated by  $s$  (spanning  $Z^p$ ), *p-coboundaries* (or, *trivial p-cocycles*) as the image, under  $s$ , of  $(p-1)$ -cochains (spanning  $B^p$ ) and the *p-th cohomology group* of  $\mathcal{G}$  as  $Z^p/B^p \equiv H^p$ . The relevance of all this technology to Lie algebra deformations can be seen by differentiating, w.r.t.  $t$ , the Jacobi identity that  $[\cdot, \cdot]_t$  must satisfy and then setting  $t = 0$ . One finds that  $\psi_1^{(2)}$  in (4) has to be a 2-cocycle. Moreover, if  $\psi_1^{(2)}$  is a coboundary, the deformation that it generates is trivial, in the sense that the resulting deformed Lie algebra is isomorphic to the original one. Referring to Fig. 1, we see that 2-cocycles span the tangent space to  $\mathcal{L}_n$  at, *e.g.*,  $Q$ , with 2-coboundaries pointing towards the  $GL(n)$  orbit of  $Q$  and non-trivial 2-cocycles outside of it. A sufficient condition then for the stability of  $\mathcal{G}$  is the vanishing of its second cohomology group  $H^2(\mathcal{G})$ . It is worth pointing out that the above is not a necessary condition. Although a non-trivial 2-cocycle may exist, obstructions originating in  $H^3(\mathcal{G})$  can render it non-integrable in which case the corresponding finite non-trivial deformation does not exist (see, *e.g.*, [16]).

It is obvious from the definition given above, that a  $p$ -cochain can always be realized as a  $\mathcal{G}$ -valued left invariant (LI)  $p$ -form on the group manifold  $G$  corresponding to  $\mathcal{G}$ , with the generators  $T_A$  now extended to LI vector fields. Denoting by  $\Pi^A$  the LI 1-forms on  $G$ , we write  $\psi^{(p)}$  as

$$\psi^{(p)} \equiv T_B \otimes \psi^B = \frac{1}{p!} \psi_{A_1 \dots A_p}{}^B T_B \otimes \Pi^{A_1} \dots \Pi^{A_p}. \quad (7)$$

Then the action of  $s$  given in (6) coincides with that of a covariant exterior derivative  $\nabla$ ,

$$\nabla(T_A \otimes \psi^A) = T_A \otimes (d\psi^A + \Omega^A{}_B \psi^B), \quad (8)$$

with the connection 1-form  $\Omega$  given by

$$\Omega^A{}_B = f_{RB}{}^A \Pi^R, \quad (\text{i.e., } \nabla_{T_A} T_B = [T_A, T_B]). \quad (9)$$

The nilpotency of  $s$  follows now from the vanishing of the curvature 2-form  $\Theta = d\Omega + \Omega^2$ , due to the Jacobi identity, while 2-cocycles are covariantly constant  $\mathcal{G}$ -valued LI 2-forms. Notice that the requirement that  $s \triangleright \psi^{(2)} = 0$ , with  $\psi^{(2)}$  as in (7), reduces to

$$f_{AR}{}^S \psi_{BC}{}^R + f_{BR}{}^S \psi_{CA}{}^R + f_{CR}{}^S \psi_{AB}{}^R + \psi_{AR}{}^S f_{BC}{}^R + \psi_{BR}{}^S f_{CA}{}^R + \psi_{CR}{}^S f_{AB}{}^R = 0, \quad (10)$$

which is, as expected, the linearized form of the Jacobi identity.

Imagine one is given all (non-zero) commutators that define a Lie algebra  $\mathcal{G}$  and is asked to strategically add to their r.h.s. (multiples of) a new, central generator  $Z$ , without violating the Jacobi identity. Assuming that this is at all possible, the resulting Lie algebra is called a *central extension* of  $\mathcal{G}$ . It is easily seen that central extensions can be considered as a particular class of deformations — one simply pretends that  $Z$  was present all the time as a  $U(1)$  factor, and got involved in the algebra, remaining central, as a result of the deformation. The general results above simplify considerably in this case. The maps  $\psi_m^{(2)}$  in (4) are valued now in the center of  $\mathcal{G}$  and, as a result, the first sum on the r.h.s. of (6) vanishes. What is left reduces, in our geometrical realization of  $s$ , to the exterior derivative  $d$ . 2-cocycles become, in this case, closed LI 2-forms, 2-coboundaries are exact LI 2-forms *with LI potential* while non-trivial 2-cocycles are closed LI 2-forms that do not have LI potential 1-form. The latter are known as *non-trivial Chevalley-Eilenberg (CE) 2-cocycles*, with obvious generalization to  $p$ -cocycles. Notice that a non-trivial CE cocycle *may* admit a potential, although not a LI one.

We wrap up our brief tour of things deformed with a couple of remarks. First, the discussion of stability above implies that the dimensions of  $\mathcal{L}_n$  and  $\mathcal{O}(P)$  must be equal in order for  $\mathcal{G}_P$  to be stable (see Fig. 1) — this would seem to single out a couple of special values of  $n$  for which stability is at all possible. However, in physical applications, one generally has to take into account additional restrictions on the structure constants (*e.g.*, Lorentz covariance) which make the balancing of dimensions more complicated and give rise to several possible values of  $n$ . Furthermore, apart from the mandatory physical restrictions, one may elect to consider only certain types of deformations, leading to a corresponding (weaker) notion of partial stability<sup>3</sup>. Second, we have dealt above, for the sake of simplicity, with ordinary Lie algebras — analogous results hold for Lie superalgebras [14], which are the ones we deal with in later sections.

As an example of a stabilizing deformation, consider the (homogeneous) Galilean algebra

$$[J_i, J_j] = \epsilon_{ijk} J_k, \quad [J_i, K_j] = \epsilon_{ijk} K_k, \quad [K_i, K_j] = 0. \quad (11)$$

Denoting by  $\Pi^i, \bar{\Pi}^i$  the LI 1-forms corresponding to  $J_i, K_i$  respectively, one easily shows that

$$\psi_1^{(2)} = \frac{1}{2} J_1 \otimes \Pi^2 \Pi^3 + \frac{1}{2} J_2 \otimes \Pi^3 \Pi^1 + \frac{1}{2} J_3 \otimes \Pi^1 \Pi^2 \quad (12)$$

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<sup>3</sup>we thank Gary Gibbons for discussions on this point.

is a non-trivial 2-cocycle (notice that the  $\mathcal{G}$ -part of  $\psi_1^{(2)}$  above does not commute with  $\mathcal{G}$  and, hence, the full expression (8) has to be used, with  $d\Pi^1 = -\Pi^2\Pi^3 + \Pi^3\Pi^2$  etc..). The resulting deformation is

$$[J_i, J_j] = \epsilon_{ijk} J_k, \quad [J_i, K_j] = \epsilon_{ijk} K_k, \quad [K_i, K_j] = -\frac{1}{c^2} \epsilon_{ijk} J_k, \quad (13)$$

where the scale of  $\psi_1^{(2)}$ , undetermined by the cocycle condition, is fixed by the experiment. This is the Lorentz algebra and, being semisimple, is stable, according to Whitehead's second lemma [11].

A second, substantially more involved, example is furnished by the passage from classical to quantum mechanics. The complexity in this case arises from the fact that the setting amenable to stability analysis is that of the infinite-dimensional Lie algebra of functions  $f(q, p)$  on phase space, with the Lie product given by the Poisson bracket — for details, see [2], while an alternative, finite-dimensional treatment is given in [12].

### 3 Quasi-invariant Actions

We lay coordinates  $\xi^i$ ,  $i = 0, \dots, p$  on the  $(p+1)$ -dimensional worldvolume  $W$  of a  $p$ -brane and embed it in  $D$ -dimensional Minkowski spacetime  $M$ , with coordinates  $x^\mu$ ,  $\mu = 0, \dots, D-1$ . The embedding, which we will generically denote by  $\phi$ , is effected by functions  $x^\mu(\xi)$ . The action is taken to be the hypervolume swept out by the brane, with metric induced by the Minkowskian  $\eta_{\mu\nu}$

$$S_p = - \int_W d^{p+1}\xi \sqrt{-\det m_{ij}}, \quad m_{ij} = \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^j} \eta_{\mu\nu}. \quad (14)$$

$M$  is a group manifold, with the group operation being ordinary translations,  $x''^\mu = x'^\mu + x^\mu$ , the generators  $X_\mu$  of which obviously commute. These leave invariant the above action — in fact, they even leave invariant the Lagrangian density, *i.e.*, the action is invariant even when surface terms cannot be ignored.

We now wish to consider the motion of our brane in superspace  $\Sigma$ . The latter is just like  $M$ , but with additional fermionic coordinates  $\theta^\alpha$ , the embedding  $\phi$  now given by functions  $\theta^\alpha(\xi)$ ,  $x^\mu(\xi)$ .  $\Sigma$  is also a (super<sup>4</sup>)group, the group operation being *supertranslations*

$$\theta''^\alpha = \theta'^\alpha + \theta^\alpha, \quad x''^\mu = x'^\mu + x^\mu + \frac{1}{2}(C\Gamma^\mu)_{\alpha\beta}\theta'^\alpha\theta^\beta. \quad (15)$$

$C$  here is the charge conjugation matrix,  $C\Gamma_\mu C^{-1} = -\Gamma_\mu^T$ . The corresponding generators  $D_\alpha$ ,  $X_\mu$  satisfy

$$\{D_\alpha, D_\beta\} = (C\Gamma^\mu)_{\alpha\beta} X_\mu, \quad (16)$$

all other commutators being zero. At this point we can establish a connection with the material of the previous section: (16) is clearly a central extension of the commutative fermionic algebra  $\{D_\alpha, D_\beta\} = 0$ , by  $X_\mu$ . Notice how this observation points to the fermionic sector of  $\Sigma$  as the fundamental one,  $M$  entering only as a stabilizer. Rewriting (16) in the dual form

$$d\Pi^\alpha = 0, \quad d\Pi^\mu = \frac{1}{2}(C\Gamma^\mu)_{\alpha\beta}\Pi^\alpha\Pi^\beta, \quad (17)$$

we identify  $(C\Gamma^\mu)_{\alpha\beta}\Pi^\alpha\Pi^\beta$  as the non-trivial CE 2-cocycle responsible for the extension.  $\Pi^\alpha$ ,  $\Pi^\mu$  here are the LI 1-forms on  $\Sigma$ , given by

$$\Pi^\alpha = d\theta^\alpha, \quad \Pi^\mu = dx^\mu + \frac{1}{2}(C\Gamma^\mu)_{\alpha\beta}\theta^\alpha d\theta^\beta. \quad (18)$$

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<sup>4</sup>we use the prefix *super* only minimally.

Notice that left invariance simply means invariance under (left) supertranslations. For this reason, the  $\Pi$ 's are the natural building blocks for a supertranslation invariant action, which takes now the form

$$S_p = - \int_W d^{p+1} \xi \sqrt{-\det m_{ij}}, \quad m_{ij} = \Pi_i^\mu \Pi_j^\nu \eta_{\mu\nu}, \quad (19)$$

where  $\phi^*(\Pi^\mu) \equiv \Pi_i^\mu d\xi^i$  is the pull-back of  $\Pi^\mu$  on  $W$ . We are not yet finished though. As mentioned already in Sec. 1,  $\kappa$ -symmetry forces the addition of a Wess-Zumino term  $S_{WZ}$  to the above action, given by the pull-back on  $W$  of a  $(p+1)$ -form  $b$  on  $\Sigma$ ,

$$S = S_p + S_{WZ}, \quad S_{WZ} = \int_W \phi^*(b). \quad (20)$$

The properties required of  $b$  are best expressed in terms of  $h = db$ :  $h$  must be a closed LI and Lorentz invariant  $(p+2)$ -form with length dimension  $p+1$ . The latter is computed by assigning length dimension 1 to  $\Pi^\mu$  and  $\frac{1}{2}$  to  $\Pi^\alpha$ . One finds

$$h = (C\Gamma_{\mu_1 \dots \mu_p})_{\alpha\beta} \Pi^{\mu_1} \dots \Pi^{\mu_p} \Pi^\alpha \Pi^\beta, \quad (21)$$

where  $\Gamma_{\mu_1 \dots \mu_p}$  is the antisymmetrized product (with unit weight) of  $\Gamma_{\mu_1}, \dots, \Gamma_{\mu_p}$ .

The natural question that arises is whether  $h$  comes from a LI potential  $b$ , so that the invariance of the total action  $S$  is maintained. The answer is a categorical no [6], which promotes  $h$  to a non-trivial CE  $(p+2)$ -cocycle.  $h$  does come from a  $(p+1)$ -form  $b$ , but the latter transforms by a total derivative under supertranslations, making  $S$  only quasi-invariant. Siegel has showed that by enlarging superspace by a new fermionic variable, a genuinely invariant action can be found in the  $p=1$  case (superstring) [18]. At the algebra level, this is just a central extension of  $\Sigma$ . The extended algebra does not admit any further extension by generators with a single Lorentz index, vectorial or spinorial, and it is only this type of generators that are relevant in the description of the superstring. We conclude that Siegel's extension of  $\Sigma$  is a stabilizing deformation, as long as one only admits generators with a single Lorentz index (this is an example of the partial stability mentioned in Sec. 1). In the case of  $p$ -branes then, it is only natural to ask the question: what is the stable form of the supersymmetry algebra (16) (or (17)), when only additional generators with  $p$  Lorentz indices are allowed?

## 4 Extensions

Our starting point is the supersymmetry algebra in its Maurer-Cartan (MC) form, Eq. (17), with  $\Pi^\mu$  rescaled so as to keep the discussion as general as possible

$$d\Pi^\alpha = 0, \quad d\Pi^\mu = a_s (C\Gamma^\mu)_{\alpha\beta} \Pi^\alpha \Pi^\beta. \quad (22)$$

We look for central extensions involving generators with  $p$  Lorentz indices, *i.e.*, for non-trivial CE 2-cocycles. We find that

$$\rho_{\mu_1 \dots \mu_p} = (C\Gamma_{\mu_1 \dots \mu_p})_{\alpha\beta} \Pi^\alpha \Pi^\beta \quad (23)$$

is the only such cocycle available. We introduce accordingly a new LI 1-form  $\Pi_{\mu_1 \dots \mu_p}$  and set its differential equal to the cocycle,

$$d\Pi_{\mu_1 \dots \mu_p} = a_0 (C\Gamma_{\mu_1 \dots \mu_p})_{\alpha\beta} \Pi^\alpha \Pi^\beta. \quad (24)$$

On the group manifold, this amounts to the introduction of a new coordinate<sup>5</sup>  $\phi_{\mu_1 \dots \mu_p}$ , so that the set  $\{\theta^\alpha, x^\mu, \phi_{\mu_1 \dots \mu_p}\}$  parameterizes now the *extended* superspace  $\tilde{\Sigma}_0$ . The relation of  $\phi_{\mu_1 \dots \mu_p}$  to  $\Pi_{\mu_1 \dots \mu_p}$

<sup>5</sup>what we mean is of course a new *type* of coordinate — there are  $\binom{D}{p}$  of them.

as well as its group law are uniquely determined by the non-LI potential for  $\rho_{\mu_1 \dots \mu_p}$ , see [5] for details. The only non-zero commutator of the resulting Lie algebra is

$$\{D_\alpha, D_\beta\} = a_s (C\Gamma^\mu)_{\alpha\beta} X_\mu + a_0 (C\Gamma_{\mu_1 \dots \mu_p})_{\alpha\beta} Z^{\mu_1 \dots \mu_p}, \quad (25)$$

where  $Z^{\mu_1 \dots \mu_p}$  denotes the generator of translations along  $\phi_{\mu_1 \dots \mu_p}$ . We now repeat the above procedure with  $\tilde{\Sigma}_0$  replacing  $\Sigma$ . We find that the introduction of  $\Pi_{\mu_1 \dots \mu_p}$ , which repaired the instability of the original algebra, has as a side effect the appearance of a new non-trivial CE 2-cocycle on the extended superspace  $\tilde{\Sigma}_0$ . Indeed, inspection of the available Lorentz tensors shows that the only two candidates for a cocycle are<sup>6</sup>

$$\rho_{\mu_1 \dots \alpha_1}^{(1)} = (C\Gamma_{\nu\mu_1 \dots \mu_{p-1}})_{\beta\alpha_1} \Pi^\nu \Pi^\beta, \quad \rho_{\mu_1 \dots \alpha_1}^{(2)} = (C\Gamma^\nu)_{\beta\alpha_1} \Pi_{\nu\mu_1 \dots \mu_{p-1}} \Pi^\beta. \quad (26)$$

For  $p = 1$ , both of these forms are closed. For  $p \geq 2$ ,  $\lambda_2 = \frac{a_s}{a_0}$  implies  $d(\rho^{(1)} + \lambda_2 \rho^{(2)}) = 0$  provided<sup>7</sup>

$$(C\Gamma^\nu)_{\alpha'\beta'} (C\Gamma_{\nu\mu_1 \dots \mu_{p-1}})_{\gamma'\delta'} = 0, \quad (27)$$

a relation which is true only for certain values of  $(D, p)$  [1]. We introduce then a LI 1-form  $\Pi_{\mu_1 \dots \alpha_1}$  and the corresponding coordinate  $\phi_{\mu_1 \dots \alpha_1}$ , obtaining the new extended superspace  $\tilde{\Sigma}_1$ <sup>8</sup>. The MC equations are augmented by

$$d\Pi_{\mu_1 \dots \alpha_1} = a_1 \left( (C\Gamma_{\nu\mu_1 \dots \mu_{p-1}})_{\beta\alpha_1} \Pi^\nu \Pi^\beta + \frac{a_s}{a_0} (C\Gamma^\nu)_{\beta\alpha_1} \Pi_{\nu\mu_1 \dots \mu_{p-1}} \Pi^\beta \right). \quad (28)$$

Notice that this new extension involves the replacement of a vectorial index by a spinorial one. Also, the appearance of  $\rho_{\mu_1 \dots \alpha_1}^{(2)}$  in  $d\Pi_{\mu_1 \dots \alpha_1}$ , implies that  $Z^{\mu_1 \dots \mu_p}$  is no longer central. Analogous remarks hold for all subsequent extensions. Every new generator introduced has one more spinorial index than the previous one, while, after each extension, the only central generator is the one introduced last, all others having acquired non-zero commutators as a result of the extensions made after the one that introduced them. The iterative procedure stops with the introduction of  $Z_{\alpha_1 \dots \alpha_p}$ , since there are no non-trivial 2-cocycles on  $\tilde{\Sigma}_p$ . The first five extensions are, from an algebraic point of view, exceptional, while for the ones that follow a pattern emerges that leads to a recursion relation. The resulting MC equations are (apart from (22), (24), (28))

$$\begin{aligned} d\Pi_{\mu_1 \dots \alpha_2} &= a_2 \left\{ (C\Gamma_{\nu\rho\mu_1 \dots \mu_{p-2}})_{\alpha_1\alpha_2} \Pi^\nu \Pi^\rho + \frac{a_s}{a_0} (C\Gamma^\nu)_{\alpha_1\alpha_2} \Pi_{\nu\rho\mu_1 \dots \mu_{p-2}} \Pi^\rho \right. \\ &\quad \left. - \frac{a_s}{a_1} (C\Gamma^\nu)_{\alpha_1\alpha_2} \Pi_{\nu\mu_1 \dots \mu_{p-2}\beta} \Pi^\beta - 8 \frac{a_s}{a_1} (C\Gamma^\nu)_{\alpha'_1\beta} \Pi_{\nu\mu_1 \dots \mu_{p-2}\alpha'_2} \Pi^\beta \right\}, \\ d\Pi_{\mu_1 \dots \alpha_3} &= a_3 \left\{ (C\Gamma^\nu)_{\alpha'_1\alpha'_2} \Pi_{\nu\rho\mu_1 \dots \mu_{p-3}\alpha'_3} \Pi^\rho + \frac{5a_1}{4a_2} (C\Gamma^\nu)_{\alpha'_1\beta} \Pi_{\nu\mu_1 \dots \mu_{p-3}\alpha'_2\alpha'_3} \Pi^\beta \right. \\ &\quad \left. + \frac{a_1}{4a_2} (C\Gamma^\nu)_{\alpha'_1\alpha'_2} \Pi_{\nu\mu_1 \dots \mu_{p-3}\beta\alpha'_3} \Pi^\beta \right\}, \\ d\Pi_{\mu_1 \dots \alpha_4} &= a_4 \left\{ (C\Gamma^\nu)_{\alpha'_1\alpha'_2} \Pi_{\nu\rho\mu_1 \dots \mu_{p-4}\alpha'_3\alpha'_4} \Pi^\rho - \frac{48a_s a_2}{5a_1 a_3} (C\Gamma^\nu)_{\alpha'_1\beta} \Pi_{\nu\mu_1 \dots \mu_{p-4}\alpha'_2\alpha'_3\alpha'_4} \Pi^\beta \right. \\ &\quad \left. - \frac{12a_s a_2}{5a_1 a_3} (C\Gamma^\nu)_{\alpha'_1\alpha'_2} \Pi_{\nu\mu_1 \dots \mu_{p-4}\beta\alpha'_3\alpha'_4} \Pi^\beta \right\}, \\ d\Pi_{\mu_1 \dots \alpha_{k+2}} &= a_{k+2} \left\{ (C\Gamma^\nu)_{\alpha'_1\alpha'_2} \Pi_{\nu\rho\mu_1 \dots \mu_{p-(k+2)}\alpha'_3 \dots \alpha'_{k+2}} \Pi^\rho + \lambda_2^{(k+2)} (C\Gamma^\nu)_{\alpha'_1\beta} \Pi_{\nu\mu_1 \dots \mu_{p-(k+2)}\alpha'_2 \dots \alpha'_{k+2}} \Pi^\beta \right. \\ &\quad \left. + \lambda_3^{(k+2)} (C\Gamma^\nu)_{\alpha'_1\alpha'_2} \Pi_{\nu\mu_1 \dots \mu_{p-(k+2)}\beta\alpha'_3 \dots \alpha'_{k+2}} \Pi^\beta \right\}, \quad k = 3, 4, \dots, \end{aligned} \quad (29)$$

<sup>6</sup>we use the index notation  $\mu_1 \dots \alpha_k \equiv \mu_1 \dots \mu_{p-k} \alpha_1 \dots \alpha_k$ .

<sup>7</sup>primed indices are understood to be symmetrized with unit weight.

<sup>8</sup>notice that the subscript of  $\tilde{\Sigma}$  counts the number of fermionic indices of the last coordinate introduced.



where

$$\lambda_2^{(k+2)} = -\frac{a_s}{a_{k+1}} \left( \frac{2}{\lambda_2^{(k+1)}} + \frac{k}{\lambda_3^{(k+1)}} \right), \quad \lambda_3^{(k+2)} = -\frac{a_s}{a_{k+1}} \frac{k+1}{\lambda_2^{(k+1)}}. \quad (30)$$

$\tilde{\Sigma}_p$  has a fibre bundle structure, with  $\Sigma$  in the base and the new coordinates  $\{\phi_{\mu_1 \dots \mu_p}, \dots, \phi_{\alpha_1 \dots \alpha_p}\}$  along the fiber. Denoting symbolically by  $D, Y$  the corresponding (classes of) generators, the Lie algebra dual to the above MC equations takes the form

$$[D, D] \sim D + Y, \quad [D, Y] \sim Y, \quad [Y, Y] = 0. \quad (31)$$

Although our way of constructing the above algebra guarantees its stability against central extensions by generators with  $p$  Lorentz indices, it is not *a priori* clear that it is stable under general deformations involving the above generators. In other words, although we know that there are no non-trivial CE 2-cocycles on  $\tilde{\Sigma}_p$ , we have not explicitly shown that there are no non-trivial 2-cocycles for the full coboundary operator  $s$ , Eq. (6). That this is indeed the case can actually be inferred from the results of [17]. We conclude that *the Lie superalgebra given by (22), (24), (28), (29), (30) is stable under deformations involving generators with  $p$  Lorentz indices.*

## 5 Applications

### 5.1 Invariant actions

We may now return to the question of finding a LI potential  $\tilde{b}$  for the  $h$  in Eq. (21). In  $\tilde{\Sigma}_p$  this is indeed possible — it takes the general form [5]

$$\tilde{b} = \sum_{k=0}^p b_k \Pi_{\mu_1 \dots \mu_k} \Pi^{\mu_1} \dots \Pi^{\mu_k}, \quad (32)$$

where the  $b_k$  are numerical constants, determined by  $h = d\tilde{b}$ . For example, for  $D = 10, p = 1$  we find Siegel's result<sup>9</sup>

$$\tilde{b} = \Pi_\mu \Pi^\mu + \frac{1}{2} \Pi_\alpha \Pi^\alpha, \quad (33)$$

while, for  $D = 11, p = 2$ , we get

$$\tilde{b} = \frac{2}{3} \Pi_{\mu\nu} \Pi^\mu \Pi^\nu - \frac{3}{5} \Pi_{\mu\alpha} \Pi^\mu \Pi^\alpha - \frac{2}{15} \Pi_{\alpha\beta} \Pi^\alpha \Pi^\beta, \quad (34)$$

in accordance with [3]. Explicit expressions for the above two cases, including the extended Lie algebras, MC equations, group law, LI vector fields and Noether currents, can be found in [5].

### 5.2 D-branes

We sketch the relevance of  $\tilde{\Sigma}_p$  in the description of  $Dp$ -branes, using as example the  $D = 10, p = 2$  IIA case. The qualitative novelty here, compared to standard  $p$ -branes, is the appearance of the Born - Infeld field  $A_i(\xi)$ , defined directly on the worldvolume. Our starting point will be the abstract free differential algebra (FDA)<sup>10</sup>

$$d\Pi^\mu = \frac{1}{2} (C\Gamma^\mu)_{\alpha\beta} \Pi^\alpha \Pi^\beta,$$

<sup>9</sup>notice that we never use the metric to raise or lower indices, so that the position of the Lorentz indices can serve to identify the generators, forms *etc...* For example,  $\Pi_\mu$  in (33) is the LI 1-form corresponding to the new coordinate  $\phi_\mu$  and has no relation to  $\Pi^\mu$  (similarly for  $\Pi_\alpha$ ).

<sup>10</sup>The term refers to an algebra generated by differential forms which is closed under the action of  $d$ , in such a way as to have  $d^2 = 0$ .

$$\begin{aligned} d\Pi^\alpha &= 0, \\ d\mathcal{F} &= \Pi^\mu (C\Gamma_\mu \Gamma_{11})_{\alpha\beta} \Pi^\alpha \Pi^\beta, \end{aligned} \quad (35)$$

where  $\mathcal{F} \equiv dA - B$  is invariant and

$$dB = -(C\Gamma_\mu \Gamma_{11})_{\alpha\beta} \Pi^\mu \Pi^\alpha \Pi^\beta. \quad (36)$$

For the above FDA  $d^2 = 0$  since, in  $D = 10$  we have  $(C\Gamma^\mu \Gamma_{11})_{\alpha'\beta'} (C\Gamma_\mu)_{\delta'\epsilon'} = 0$ . One looks for non-trivial  $(p+2)$ -cocycles  $h$ , constructed from the above forms, with length dimension  $p+1$ , so as to match that of the kinetic Lagrangian. Such cocycles exist only for  $p \leq 8$  and even, *i.e.*, precisely for those values for which  $Dp$ -branes of type IIA are known to exist. For  $p = 2$ ,  $h$  takes the form

$$h = (C\Gamma_{\mu\nu})_{\alpha\beta} \Pi^\mu \Pi^\nu \Pi^\alpha \Pi^\beta - (C\Gamma_{11})_{\alpha\beta} \Pi^\alpha \Pi^\beta \mathcal{F}, \quad (37)$$

while its LI potential follows (apart from the  $\mathcal{F}$  term) from an appropriate dimensional reduction of (34),

$$\tilde{b} = \frac{2}{3} \Pi_{\mu\nu} \Pi^\mu \Pi^\nu + \frac{4}{3} \Pi_\mu \Pi^\mu \Pi - \frac{2}{15} \Pi_{\alpha\beta} \Pi^\alpha \Pi^\beta - \frac{3}{5} \Pi_{\mu\alpha} \Pi^\mu \Pi^\alpha - \frac{3}{5} \Pi_\alpha \Pi \Pi^\alpha - 2\Pi \mathcal{F}. \quad (38)$$

Moreover,

$$d\left(\frac{1}{2} \Pi^\alpha \Pi_\alpha - \Pi^\mu \Pi_\mu\right) = (C\Gamma_\mu \Gamma_{11})_{\alpha\beta} \Pi^\mu \Pi^\alpha \Pi^\beta, \quad (39)$$

which shows that one may set, on the extended superspace,

$$\mathcal{F} = \frac{1}{2} \Pi^\alpha \Pi_\alpha - \Pi^\mu \Pi_\mu. \quad (40)$$

Finally, the worldvolume field  $A$  may be identified with the pull-back, on  $W$ , of the following 1-form

$$A = \varphi_\mu dx^\mu + \frac{1}{2} \varphi_\alpha d\theta^\alpha, \quad (41)$$

defined also on the extended superspace. Similar results hold for the  $D = 11$   $M5$ -brane — the details may be found in [5].

## 6 Epilogue

Our aim has been to argue that Lie (super)algebra stability is a sensible criterion for the soundness of a physical theory. To the standard list of examples, we may now add the results of [5], where the natural emergence of extended superspaces in the description of extended objects leads to the resolution of a number of problems arising in the conventional formulation. Assuming a (even partially) convinced reader, we take a further step in this direction proposing that the recent proliferation of non-commutative spaces, as the arenas for physical phenomena, should be examined in the light of stability considerations. In particular, the form of the spacetime coordinate commutation relations could be inferred, or at least narrowed down, by demanding the stability, under suitable restrictions, of the resulting Lie algebra, much in the spirit of the Poincaré algebra analysis in [12]. The novelty that arises, in view of our results, is the possibility to include the additional coordinates of the extended spaces encountered. We plan on pursuing this theme in a forthcoming publication [4].

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